Time-Domain Simulation of Electromagnetic Field Using a Symplectic Integrator

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***Abstract*—A new high-order-accurate method for time-domain simulation of a electromagnetic field is proposed. This method is analogous to the one used in classical mechanics for continuous system analysis. The symplectic integrator in classical mechanics is utilized to calculate the time dependence of the field. Using the same grid spacing and time step, it is demonstrated that the proposed method is more accurate than the conventional finite-difference time-domain (FDTD) method for a two-dimensional problem. This method is also applicable to full-vectorial Maxwell's equations in three dimensions.**

***Index Terms*—Electromagnetic fields, FDTD methods, numerical methods, symplectic integrator, time-domain analysis.**

I. Introduction

THE finite-difference time-domain (FDTD) method has been used extensively for electromagnetic field simulation [1], [2]. A problem with the FDTD method is its demand for vast computational resources. One solution is to use a more accurate scheme than the conventional one to reduce the required memory size. Higher order differencing schemes have been proposed [3]–[5] and their performance for practical applications have been studied [6]–[9]. New higher order differencing schemes are also attracting attention [9], [10].

This letter introduces a new time-domain method for deriving high-order-accurate schemes. We treat the electromagnetic field as a continuous system in classical mechanics [11]. The treatment enables us to use a numerical method for a Hamiltonian system known as the symplectic integrator method [12]–[18]. It is a Runge–Kutta method which preserves the symplectic structure of the phase space. Its usefulness has been verified [12], [13], [16], [18]. Our results of elementary simulations of the electromagnetic field show that the symplectic integrator method is promising.

II. Formulation

We outline the explicit symplectic integrator for a system governed by the Hamiltonian $H_0 = V(\mathbf{q}) + T(\mathbf{p})$, in which $\mathbf{q}$ is generalized coordinates and $\mathbf{p}$ is momenta conjugate to $\mathbf{q}$ [18]. The nth-order symplectic integrator $S_n(\Delta t)$ for a time-step Δt is

$$S_n(\Delta t) = \prod_{i=1}^{n} S_T(c_i \Delta t) S_V(d_i \Delta t) + O((\Delta t)^n) \qquad (1)$$

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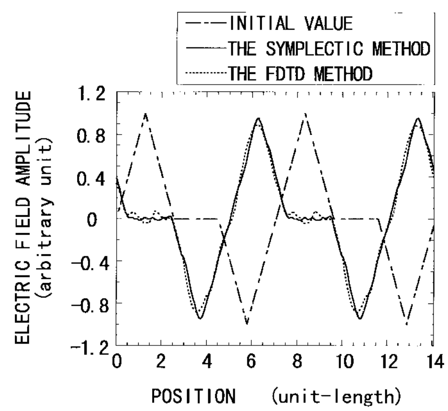

Fig. 1. Electric field profiles simulating the propagation of the two-dimensional TE-mode plane wave by means of the fourth-order symplectic method [Scheme (I) in the text] and the FDTD method. The simulated region is a square whose sides are ten unit-length and are parallel to the x or y axis. The wave propagates diagonally. Permittivity is 1.0 and permeability is 1.0 in the square. Thus, the phase velocity of the wave is 1.0 unit-length per unit-time. The boundary condition is periodic. In both methods, the plane lattice is square and its spacing is 0.05 unit-length and the time-step is 0.025 unit-time. The dot-dash line indicates the initial electric field profile on the diagonal line of the simulated region. The solid line indicates the profile after 20 000 time-step calculations by the symplectic method. The dotted line indicates the profile after 20 000 time-step calculations by the FDTD method.

where $S_T(c_i\Delta_t)$ and $S_V(d_i\Delta t)$ are symplectic mappings from $(\mathbf{q}, \mathbf{p})$ to $(\mathbf{q}', \mathbf{p}')$ as

$$S_T(c_i\Delta t): \mathbf{q}' = \mathbf{q} + c_i\Delta t\left(\frac{\partial T}{\partial \mathbf{p}}\right), \quad \mathbf{p}' = \mathbf{p}, \qquad (2)$$

$$S_V(d_i\Delta t): \mathbf{q}' = \mathbf{q}, \quad \mathbf{p}' = \mathbf{p} - d_i\Delta t\left(\frac{\partial V}{\partial \mathbf{q}}\right). \qquad (3)$$

Parameters c_i and d_i are determined, so that $S_n(\Delta t)$ can approximate the time evolution of the system with the accuracy of the order $(\Delta t)^n$. They are $c_1 = c_2 = 1/2$, $d_1 = 1$, $d_2 = 0$, for $S_2(\Delta t)$ [14] and $c_1 = c_4 = 1/\{2(2 - 2^{1/3})\}$, $c_2 = c_3 = (1 - 2^{1/3})/\{2(2 - 2^{1/3})\}$, $d_1 = d_3 = 1/(2 - 2^{1/3})$, $d_2 = -2^{1/3}/(2 - 2^{1/3})$, $d_4 = 0$ for $S_4(\Delta t)$ [16].

In our formulation, the Hamiltonian, H, of the electromagnetic field is

$$H = \int \left\{ \frac{\boldsymbol{\pi}^2}{2\varepsilon} + \frac{(\nabla \times \mathbf{A})^2}{2\mu} - \mathbf{j} \cdot \mathbf{A} \right\} dx\, dy\, dz \qquad (4)$$

where

x, y, and z	orthogonal space coordinates;
ε	permittivity;
μ	permeability;

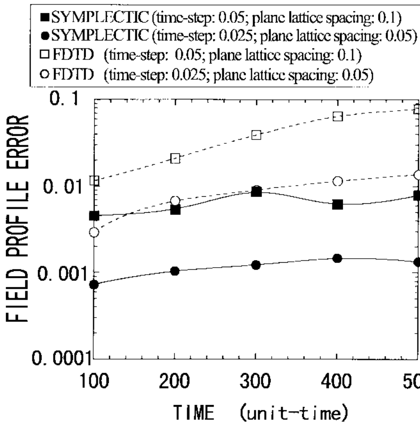

Fig. 2. Field profile errors of the fourth-order symplectic method [Scheme (I) in the text] and the FDTD method as functions of time t. The simulation conditions are the same as those in Fig. 1, except for the calculation time-step and the plane lattice spacing.

$\mathbf{A}$	vector potential;
$-\boldsymbol{\pi}$	electric displacement;
$\mathbf{j}$	current density.

We consider $\mathbf{A}$ as the generalized coordinates and $\boldsymbol{\pi}$ as the generalized momenta conjugate to $\mathbf{A}$. The equations of motion are

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\boldsymbol{\pi}}{\varepsilon} \tag{5}$$

$$\frac{\partial \boldsymbol{\pi}}{\partial t} = -\nabla \times \left(\frac{\nabla \times \mathbf{A}}{\mu}\right) + \mathbf{j}. \tag{6}$$

We use a gauge in which the scalar potential is zero. Thus (5) and (6) are equivalent to Maxwell's equations when the following equation is valid over the entire simulated region with the initial condition

$$\nabla \boldsymbol{\pi} = -\rho, \tag{7}$$

where ρ is the charge density.

Here, we introduce two spatial discretizing procedure for the system. One procedure is the application of the finite-difference method to (6). We apply it to the two-dimensional TE-mode condition with constant μ in the simulated region and without current density and make Scheme (I). When the electric field vector is along the z axis, the right side of (6) is reduced to $\nabla^2 A_z(x,y,t)/\mu$, where ∇^2 is the two-dimensional Laplacian. The Laplacian is descritized using the fourth-order approximation on the square lattice such that

$$\begin{aligned}(\nabla^2 f)_{i,j} \approx \frac{1}{60h^2}\{&52(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1})\\ &- (f_{i+2,j} + f_{i-2,j} + f_{i,j+2} + f_{i,j-2})\\ &+ 16(f_{i+1,j+1} + f_{i-1,j+1} + f_{i+1,j-1} + f_{i-1,j-1})\\ &- 2(f_{i+1,j+2} + f_{i-1,j+2} + f_{i+1,j-2} + f_{i-1,j-2}\\ &+ f_{i+2,j+1} + f_{i-2,j+1} + f_{i+2,j-1} + f_{i-2,j-1})\\ &- 252 f_{i,j}\}\end{aligned} \tag{8}$$

where

f	function of (x,y);
$f_{i,j}$	f value on the lattice point (x_i, y_j);
$\nabla^2 f_{i,j}$	$\nabla^2 f$ value on the lattice point (x_i, y_j);
h	plane lattice spacing.

In Scheme (I), the integrator $S_4(\Delta t)$ is used.

Scheme (II) is based on another procedure in which the Hamiltonian (4) is discretized by the shape functions of the finite-element method. We use the linear serendipity element for cubic lattice [19] to integrate the term containing $\nabla \times \mathbf{A}$. The integration of the term containing $\mathbf{j}$ or $\boldsymbol{\pi}$ is substituted by the summation over the cubic lattice points. $S_2(\Delta t)$ is applied to the spatially discritized Hamiltonian.

The scheme, to which the integrator $S_n(\Delta t)$ is applied, approximates all Maxwell's equations with accuracy of the order $(\Delta t)^n$. Thus, the charge conservation condition is satisfied with the same order of accuracy.

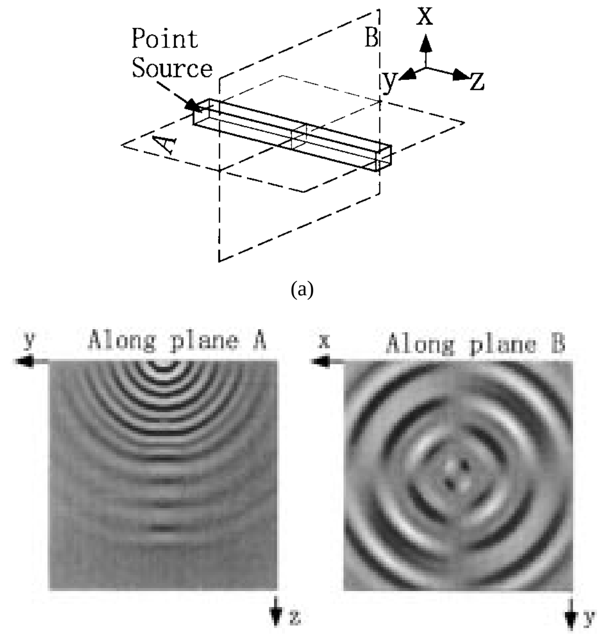

Fig. 3. Full vectorial three-dimensional simulation by means of the symplectic method [Scheme (II) in text]. (a) Schematic drawing of the simulated structure. The rectangular prism is a waveguide whose permittivity is 10% larger than the environment. The sinusoidal wave, whose vector potential is polarized in the x-direction, radiates from the point source. The space lattice is cubic and its spacing is one-tenth of the wavelength. The time increment is one-twentieth of the wave period. (b) The amplitude of the x component of the vector potential in the y-z plane [i.e. plane A]. (c) The amplitude (x 40) of the y component of the vector potential in the x-y plane (i.e., plane B).

III. Numerical Results and Discussion

We compare the long-term simulation accuracy of Scheme (I) to that of the FDTD method in terms of the initial value problem under the periodic boundary condition. In both calculation methods, initial values are discretized from the initial values of a continuous model, in which a plane wave with a nonsinusoidal electric field profile propagates without any changes in the profile. Fig. 1 shows the initial profile of the electric field on the diagonal line in the simulated square region, the profile after Scheme-(I) calculation for 20 000 time steps, and that obtained by the FDTD method for the same time steps. The Scheme-(I) calculation preserves the initial field profile better than the FDTD method. Scheme (I) takes almost double the computation time of the FDTD method on a HP-9000 work station.

Fig. 2 shows field profile errors as a function of time t. In this example, the error of Scheme (I) with the lattice spacing of 0.1 unit-length is almost the same as that of the FDTD method with the lattice spacing of 0.05 unit-length. Alternatively, using Scheme (I), the total grid number is reduced without compromising the accuracy of the calculation results.

From our experience, we found Scheme (I) is at least as stable as the FDTD method. Scheme (I) is applicable to the problem in which ε is not constant in space. This scheme is stable under the perfect conductor boundary condition. When Scheme (I) is applied in the unbounded domain, the introduction of losses near the lattice boundary prevents the reflection of outgoing waves at the boundary.

Fig. 3 shows a full vectorial calculation by Scheme (II). The qualitative features of the wave propagation along the waveguide is well simulated by this scheme. This result suggests that the symplectic integrator is applicable to the three-dimentional problem. However, the field profile error under the same accuracy test as in Fig. 2 are larger than that of the FDTD method. At this time, it appears that the dispersion error of Scheme (II) is too large to simulate the field profile, whose wavenumber range is as broad as the initial field profile shown in Fig. 1. We anticipate that, through further research, we can improve the accuracy of the three-dimentional scheme.

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